Phase transitions in an Ising model on a Euclidean network

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A one-dimensional network on which there are long-range bonds at lattice distances $l>1$ with the probability $P(l)\propto l^{-\delta}$ has been taken under consideration. We investigate the critical behavior of the Ising model on such a network where spins interact with these extra neighbors apart from their nearest neighbors for $0\leqslant\delta<2$. It is observed that there is a finite temperature phase transition in the entire range. For $0\leqslant\delta<1$, finite-size scaling behavior of various quantities are consistent with mean-field exponents while for $1\leqslant\delta\leqslant 2$, the exponents depend on δ. The results are discussed in the context of earlier observations on the topology of the underlying network.

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I. INTRODUCTION

During the last few years there has been a lot of activity in the study of networks once it was realized that networks of diverse nature exhibit some common features in their underlying structure. The Watts-Strogatz (WS) model [1] was proposed to simulate the small-world feature of real networks (i.e., the property that the networks have a small diameter as well as a large clustering coefficient). In this model, the nodes are placed on a ring and each node has a connection to a finite number of nearest neighbors initially. The nearest-neighbor links are then rewired with a rewiring probability p_r to form random long-range links. This model displays phase transitions from a regular to a small world to a random network by varying the single parameter p_r representing disorder. The regular to small-world network transition was shown to take place at $p_r\to 0$. In a slight variation of the WS model, the addition-type network was considered in which random long-range bonds are added with a probability p, keeping the nearest-neighbor links undisturbed. Phase transition to a small-world network was again observed for $p\to 0$ [2,3].

Phase transitions in networks can also be driven by factors other than the above kind of disorder. In many real-world networks, the linking probability is dictated by factors such as Euclidean distances separating them, aging, etc. It is possible to achieve phase transitions in the network by tuning the parameters governing such factors. These networks are indeed not just theoretical concepts. Many real-world networks such as the Internet at the router level, transport networks, power grid networks, and even collaboration networks are indeed described on a Euclidean space in which the geographical locations of the nodes play an important role in the construction of the network [4–7]. Similarly, the aging factor is important in social networks and citation network [3,8,9] where it is found that linking with older nodes is generally less probable.

In the theoretical modeling of Euclidean networks [10], it is usually assumed that two nodes separated by a Euclidean distance l are connected with the probability $P(l)$, which follows a power-law variation, i.e.,

$$P(l)\sim l^{-\delta}. \tag{1}$$

In one dimension, where the nodes are placed on a ring, the typical networks generated for different values of δ and the corresponding adjacency matrix are shown graphically in Fig. 1. Growing networks on Euclidean spaces where the linking probability is modified with such a probability factor have also been considered recently [11]. In a real situation, however, $P(l)$ may not have such a well-defined behavior [5,7].

A variation of $P(l)$ in the above form was first studied by Kleinberg [12] on a two-dimensional plane with the aim to find out how navigation in the network depends on the parameter δ. Later, a number of other properties of networks (both static and dynamic) with such a probabilistic attachment have been studied [10,13–17].

In the present study, we have reconsidered a static one-dimensional Euclidean network (OEN) with connection probability at distance l given by Eq. (1). We are interested in the phase transitions, which can be achieved by varying the parameter δ. Although this issue has been addressed earlier [13–16], there remain some questions about the exact nature of phase transitions in the region $0<\delta<2$. We have tried to resolve the problems by taking a different approach here. In the earlier studies, the topological features such as the diameter, shortest paths, and the clustering coefficient of the network had been studied. Here we have considered Ising spins on the nodes of such a network. The critical behavior of this system is expected to reflect the nature of the network at different values of δ indirectly. That the Ising model undergoes phase transitions on the addition-type WS network is an established fact [18–20]. It has also been shown that in the small world phase the addition-type WS has a mean-field nature [21,22].

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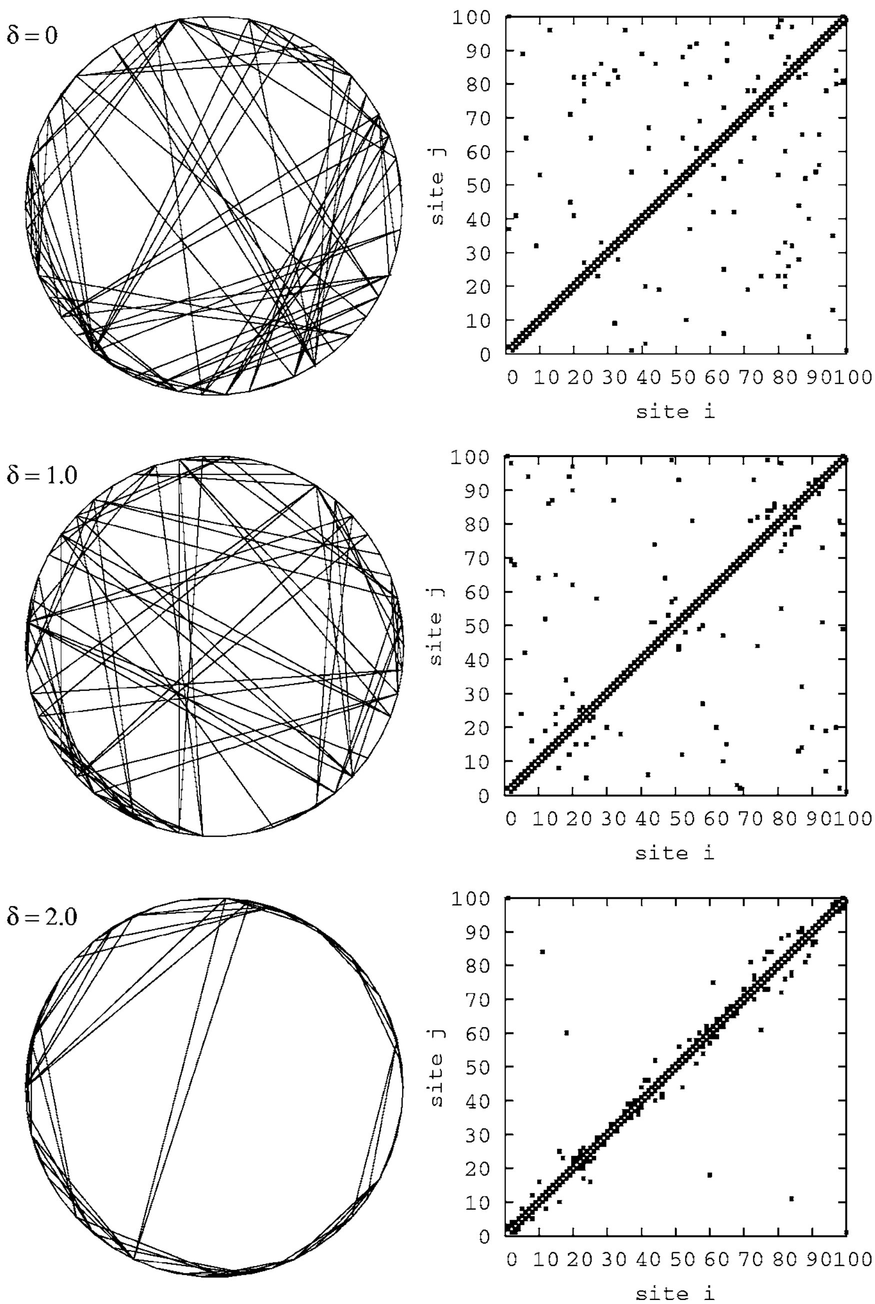

FIG. 1. Structures of the networks (left panel) and the corresponding adjacency diagrams (right panel) for δ=0, 1.0, 2.0 (top to bottom) in the model for a system of size N=100.

In Sec. II, we have reviewed the available results of the Euclidean networks and also justified the necessity of the present investigations. In Sec. III, studies of Ising models on networks have been briefly reviewed. Our results of the study of the Ising model on a Euclidean network is given in Sec. IV. Finally, in Sec. V, a summary and conclusions are given.

II. NETWORK ON EUCLIDEAN SPACE

The OEN, with the connection probability given by Eq. (1), is equivalent to the addition-type WS model at δ=0 when the addition of long-range links can take place at any length scale. The choice of the number of long-range neighbors in this OEN should be such that it allows a phase transition at δ=0. In the addition-type WS model, this transition for a system of N nodes is achieved for $p=1/N$ (implying $p\rightarrow 0$ in the thermodynamics limit). Connection probability $p=1/N$ implies a total number of N long-range edges in the system or the existence of one long-range bond per node on an average. Therefore in order to obtain a phase transition in the OEN, it is sufficient to keep N long-range bonds. For large values of δ, the nodes in the network have short-range connections only and therefore it behaves as a regular network [13–15]. Thus there should exist at least one phase transition in this network.

In all, there could be four kinds of behavior of this network: (a) regular: when the network is like a short-ranged one-dimensional system, (b) finite dimensional: when the network behaves as a system with effective dimensionality greater than one but still short ranged, (c) small world, and

(d) random. A few studies in which these phase transitions have been investigated have not been able to give a concrete or unique answer. The network behaves as a regular network above $\delta=2$; this is obtained in all the earlier studies. The interesting region is $0 \leqslant \delta \leqslant 2$ as here the network no longer behaves like a one-dimensional system. In Ref. [13], the behavior of the OEN was studied by releasing a random walker on the network. The results indicated that the walker has the same behavior as on a small-world network for all $\delta < 2$.

To detect whether a network has small-world behavior, one can calculate the shortest distances separating two nodes, take the average, and analyze its behavior with the system size. The largest of these shortest paths is called the diameter of the network. The average shortest path and diameter of a network are expected to have the same scaling behavior. Sen and Chakrabarti [14] studied the diameters of the OEN while the shortest paths were calculated by Moukarzel and de Menezes [15] giving contradictory results. In Ref. [14], it was found that the diameter behaved as $\ln N$ on chains of size N for all $\delta < 2$ and hence it was concluded that small-world behavior occurs for $0 \leqslant \delta \leqslant 2$ while in Ref. [15], it was argued that small-world behavior occurs only for $\delta < 1$. In the region $1 < \delta < 2$, according to Ref. [15], the shortest distances scale as N^{θ} with the value of the shortest path exponent θ $(0 < \theta < 1/2)$ depending on δ. The system sizes considered in Ref. [15] were much larger compared to those in Ref. [14] and the discrepancy of the results could be accounted for by this fact. It must also be mentioned here the above results are for $pN=1$; results for other values of p have also been considered in these studies.

Random and small-world phases can be distinguished by the clustering coefficient, which remains finite in the latter. The estimate of the clustering coefficients of the present network [16] showed that below $\delta=1$ it vanishes. This was considered to be a signature of the network being random for $\delta < 1$ and based on the findings of Ref. [14], it was concluded that the region $0 \leqslant \delta < 2$ was equally divided into a random (for $\delta < 1$) and a small-world (for $\delta > 1$) phase. But this is by no means foolproof, as in this particular network the clustering coefficient is bound to be small close to $\delta=0$ as the number of nearest neighbors is small when δ is small. (In the original WS model, the clustering coefficient remained high for finite values of $p_r \neq 1$ only when the number of nearest neighbors was at least four to begin with.) Thus it may not indicate a random to small-world phase transition at $\delta=1$ as conjectured in Ref. [16], it could, in principle, correspond to the small world to a finite dimensional network as found in Ref. [15]. That there could be two transitions, one at $\delta=1$ and the second at $\delta=2$, was also supported by simple scaling arguments for the average bond lengths[16].

It has been mentioned in the beginning of this section that the total number of random long-range bonds per site has been kept equal to 1 on an average such that $p=1/N$. This is also consistent with the fact that such a choice will keep the probability normalized. However, in an analytical approach there will be a problem at $\delta=1$ where

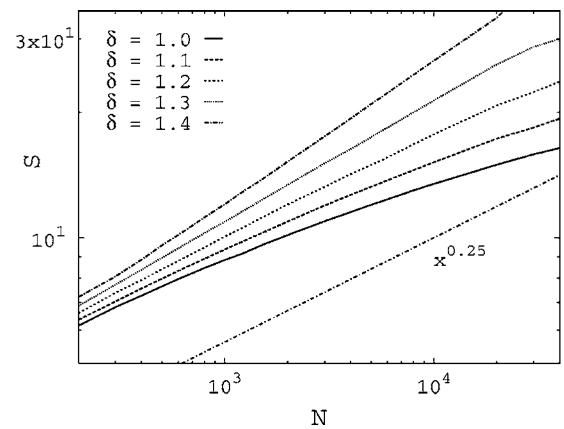

FIG. 2. Behavior of shortest path S with system size N.

such a normalization is not possible. This issue has been discussed in Ref. [13] and does not cause a problem in numerical calculations.

Navigation or searching on small-world networks are known to give rise to shortest paths, which do *not* behave as $\ln(N)$ but rather have sublinear variations with N except for special points: this was first detected in Ref. [12] for two-dimensional lattices and later confirmed for the WS model [23]. In the present one-dimensional Euclidean model this was again demonstrated in Ref. [17].

The more recent results [15,16] indicate that there is indeed a phase transition at $\delta=1$ and that the network behaves as a small world only for $\delta < 1$. In the present work we want to confirm this using a completely different approach. If the network's behavior is that of a small world it should be reflected in the critical exponents of the Ising model, which would assume mean-field values. On the other hand, if it behaves like a finite dimensional system with effective dimension greater than one, a phase transition will be observed with critical exponents assuming values different from the mean-field ones (note that the lower critical dimension of the Ising model is 1).

Another interesting point which can be studied by considering the Ising model on this network is the relation between the shortest path exponent θ and the effective dimension of the network for $1 < \delta < 2$. If one interprets θ as the inverse of some effective dimension d_{eff} then, according to Ref. [15], for some values of $\delta > 1$, d_{eff} is actually greater than 4. In fact, we have used the burning algorithm [24] to find out independently the shortest paths between *all* pairs of nodes in a network of size N and found out that indeed θ has values below 0.25 up to $\delta \simeq 1.2$ (Fig. 2). Our estimates of θ are slightly larger than those obtained in Ref. [15], which may be due to the smaller system sizes considered, but still we find $\theta < 0.25$ over a considerable range of values of $\delta > 1$. The question that naturally arises is whether the Ising model shows mean-field behavior here as the effective dimension is higher than the upper critical dimension, i.e., does it have a mean-field behavior even though the network is not a small world?

III. ISING MODELS ON NETWORKS

Ising models with long-range neighbors have been considered in different contexts in many earlier studies. It is well known that the short-ranged Ising model on a one-dimensional lattice does not show any phase transition while interaction with all the neighbors gives rise to a mean-field-like phase transition. The occurrence of the random long-ranged bonds also enables a phase transition [18,19,21,22,25–27].

In an analytical study of the Ising model on a WS-type network, it was assumed that there are two nearest neighbors and additional long-range neighbors occur randomly with probability $p=\rho/N$ [19] where ρ is a parameter. Phase transition occurs here for $\rho \geqslant 1$ implying that the addition of one long-range bond to each node (on an average) could enhance a phase transition. In the numerical studies also, the addition of one random long-range neighbor per site enables a phase transition [25]. The fact that the Ising model on these networks show a phase transition even when the additional long-range connections are finite in number [i.e., $O(N)$] compared to that of the infinite range model [where there are $O(N^2)$ number of bonds] is evidence of the key role played by the randomness in these networks [26].

Another kind of long-range Ising model has been considered in which the interactions decrease with the distance as a power law, i.e, the Hamiltonian is given by [28]

$$H = -\Sigma_{ij} J_{ij} S_i S_j, \tag{2}$$

where $J_{ij} \sim 1/r_{ij}^{\mu}$; r_{ij} is the distance separating the spins at sites i and j. In the fully connected model, phase transitions occur at the values $\mu=3/2$ and $\mu=2$; for $\mu<3/2$, the system is mean-field-like while for $3/2<\mu<2$ it is like a finite-dimensional lattice and for $\mu>2$, it behaves like a one-dimensional system with no phase transition. On a small-world network, the interactions exist randomly among neighbors, maintaining the form of J_{ij} as above. However, there is no phase transition displayed by this model [29] except at $\mu=0$ where it coincides with the WS model.

As mentioned earlier, the critical behavior of the Ising model on a WS addition-type network is mean-field type. The intriguing feature is the validity of the finite-size scaling analysis in the mean-field regime. Moreover, it has been observed [25] that the data collapse in the finite-size scaling analysis can be achieved when $|T-T_c|$, the deviation from the critical temperature, is scaled by the factor $N^{-1/\bar{\nu}}$ in the scaling argument with $\bar{\nu}=2$. Interpreting $\bar{\nu}$ as νd, where ν is the critical exponent for the correlation function and d is the effective dimensionality of the system, could imply that one has effectively a system with $d=4$ with the mean-field value of $\nu=1/2$. This effective dimension is identical to the upper critical dimension of the Ising model.

Phase transition of the Ising model has also been observed in scale-free networks [30,31] where the transition temperature diverges with the system size. Several other aspects of Ising models on networks have been considered recently, for example, the self-averaging property of an Ising model on networks [26], quenching dynamics [32], etc., which are not directly related to the content of the present paper.

IV. ISING MODEL ON EUCLIDEAN NETWORK

We have considered an Ising model on a finite chain of length N with a periodic boundary condition. The nodes are assigned position coordinates $1,2,\ldots,N$ along the chain. Each node is connected to its two nearest neighbors. In order to generate the long-range bonds, two nodes are selected randomly. If $l>1$ is the distance separating them, they are connected with probability $P(l)$ as given by Eq. (1). The process is repeated $N/2$ times generating N long-range bonds (each bond is counted twice so that there are N long-range bonds in the system). This ensures that there is one long-range neighbor for each site on an average. As has been mentioned in the previous section, this is sufficient to achieve a phase transition. For each realization of the network, the dynamical evolution from a uniform state (all spins up) was allowed following a Metropolis algorithm for different temperatures T.

We have computed the following quantities on this network after it reaches equilibrium:

(1) *Magnetization per spin* $m=\Sigma_i S_i/N$.

(2) *Binder cumulant* $U=1-\frac{\langle m^4\rangle}{3(\langle m^2\rangle)^2}$.

(3) *Susceptibility per spin* is calculated from the fluctuation of the order parameter:

$$\chi = \frac{N}{K_B T}[\langle M^2\rangle - \langle M\rangle^2],$$

where M is the total magnetization, T is the temperature, and K_B is the Boltzmann's constant.

(4) *Specific heat per spin* is calculated from the fluctuation of the energy of the system as

$$c = \frac{N}{(K_B T)^2}[\langle E^2\rangle - \langle E\rangle^2],$$

where E is the energy of the system.

A. Results

From the intersection of U for different N in the plot of U vs T we estimate T_c and a data collapse is obtained by plotting U vs $(T-T_c)N^{1/\bar{\nu}}$. With this value of $\bar{\nu}$, one can now estimate the exponents β, γ, and α, the critical exponents for the order parameter m, susceptibility χ, and specific heat c, respectively, using finite-size scaling. Figures 3 and 4 show the data collapse for these quantities for δ=0.6 and δ=1.4. Similar results have been obtained for other values of δ.

1. Transition temperature

As expected we find a transition temperature T_c, which decreases with δ (Fig. 5). T_c varies very slowly in the region $0<\delta<0.5$ and much faster for higher values of δ.

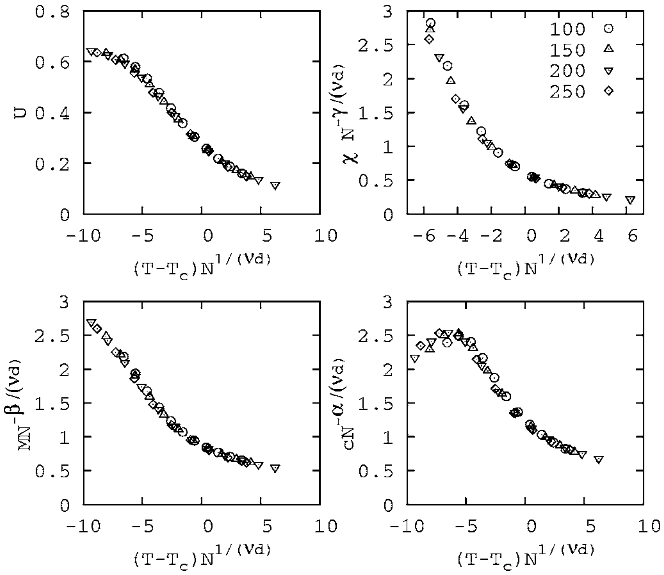

FIG. 3. Data collapse for different system sizes N=100, 150, 200, 250 for U, χ, M, and c, respectively, for δ=0.6 ($\bar{\nu}=\nu d$).

2. Critical exponents

From the finite-size scaling analysis, we obtain the critical exponents $\bar{\nu}$, β, γ, and α (Figs. 6 and 7). We find that $\bar{\nu}$ is equal to 2 for the entire range $0 \leqslant \delta < 1$. However, for $\delta > 1$ we find that $\bar{\nu}$ decreases and then rises again.

The magnetization exponent β is equal to 0.5 for $\delta < 1$, while for $\delta \geqslant 1$, it decreases with δ. The susceptibility exponent γ is close to 1 for $\delta < 1$ and increases with δ beyond δ=1. The specific-heat exponent α remains constant at a value close to 0.1 for $\delta < 1$ and appears to deviate from this value as δ is made larger than 1. It becomes small as δ approaches 2.

The behavior of all the quantities is consistent with the fact that for $\delta < 1$, there is a mean-field behavior of the Ising

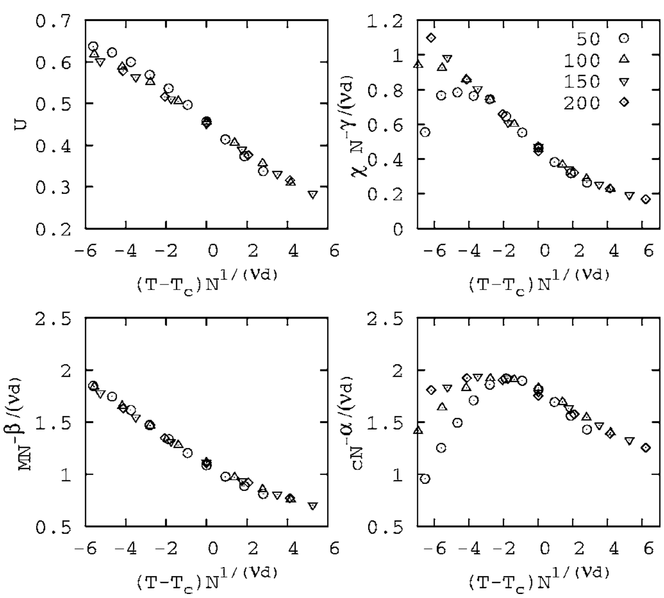

FIG. 4. Data collapse for different system sizes N=50, 100, 150, 200 for U, χ, M, and c, respectively, for δ=1.4 ($\bar{\nu}=\nu d$).

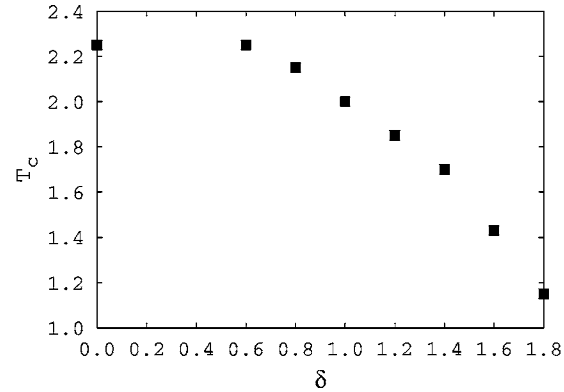

FIG. 5. Variation of critical temperature T_c with δ. The Errors in this and the next two figures are of the order of the size of the data points.

model (consistent with the available solution for the random long-range bond small-world network [27]) and for $\delta > 1$, its behavior is like a finite-dimensional lattice with effective dimension greater than one.

As has been argued for δ=0, here also one can interpret $\bar{\nu}$ as νd, where d is the effective dimension, such that it satisfies $\bar{\nu}=2-\alpha$. Since below d=4 α is nonzero, $\bar{\nu}$ becomes less than 2 at $\delta > 1$. Theoretically, the behavior of α is nonmonotonic as a function of dimensionality d as d varies from 4 to 1; α is zero for both d=4 and d=2 and this behavior is reflected in the fact that $\bar{\nu}$ again rises to 2 at around δ =1.8. However, our result for α does not show this nonmonotonic behavior very accurately, the reason possibly being that the magnitude of α is small $[O(0.1)]$ even when it is nonzero and is difficult to estimate accurately in a numerical study.

The mean-field value for γ is 1.0, which comes out to be slightly lower in our estimate and also, the value of α is higher than the corresponding 0, but the scaling relation $\alpha+2\beta+\gamma=2$ is obeyed with a high degree of accuracy. Although there are deviations of α and γ from the mean-field

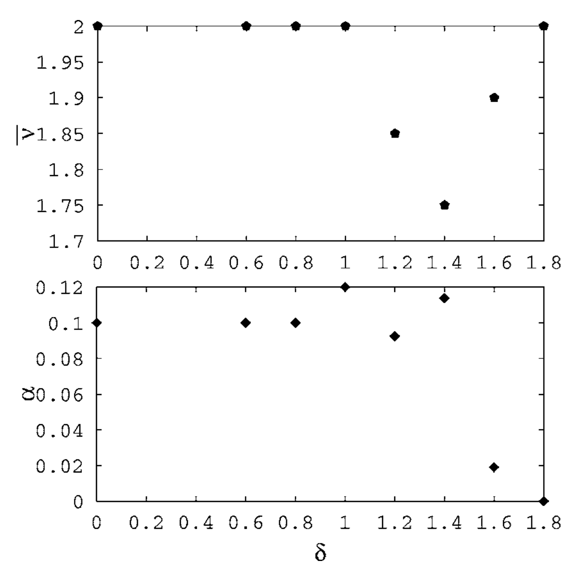

FIG. 6. Variation of $\bar{\nu}$ and specific heat exponent α with δ.

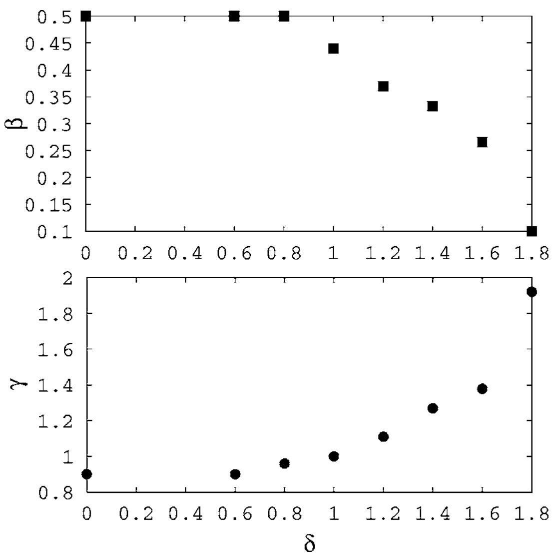

FIG. 7. Variation of magnetization exponent β and susceptibility exponent γ with δ.

values, the fact remains that they show no variation over the region $0 \leq \delta < 1$, indicating that the critical behavior for $\delta < 1$ is identical to that for $\delta = 0$, which is known to be mean-field-like.

B. Effective dimension

Our results confirm that the network behaves as a finite dimensional system in the region $1 < \delta < 2$. However, the finite-size scaling analysis of various quantities in the Ising model does not allow the estimation of d and ν independently. It may be expected that the behavior of the average shortest path should give us some idea about the effective dimension. However, as we argue below, it is not possible. The estimate of the exact average shortest path is consistent with the mean-field picture for $\delta < 1$ as it shows a logarithmic increase with N (small-world effect). As mentioned before, for $\delta > 1$, it has a power law increase N^{θ} and we have a region where $\theta < 0.25$. So if one interprets θ as an inverse dimension, the dimensionality is greater than 4 here. Even then, the Ising model does not show mean-field behavior here. From this we conclude that although there is an effective dimension coming out from the topological behavior of the network, it does not exactly act as the spatial dimension as far as the physics of the Ising model is concerned.

We adopt a different approach to estimate ν and d, which works well for δ close to 1. The behavior of the critical exponents is fully consistent with the fact that the effective dimensionality d decreases as δ is made larger than 1 (e.g., the value of γ increases while that of β decreases). In order to find out d, we use the epsilon expansions of α, β, and γ to first order, each of which gives an independent estimate of $\epsilon = 4 - d$:

$$\epsilon = 3(1 - 2\beta),$$

$$\epsilon = 6(\gamma - 1),$$

TABLE I. The effective dimension and comparative values of νd from ϵ expansion (ee) and $\bar{\nu}$ from finite-size scaling (fss) analyses.

δ	ν_{eff}	d_{eff}	νd (ee)	$\bar{\nu}$ (fss)
1.2	0.56	3.34	1.85	1.85
1.4	0.59	2.90	1.72	1.75
1.6	0.61	2.74	1.66	1.9
1.8	0.72	1.36	0.98	2.0

$$\epsilon = 6\alpha.$$

Ideally, all three estimates should give a similar value of ϵ, but here we have used the expansions up to linear terms only and the estimates may not agree very well (especially for the estimate from α and to some extent, that from γ). It is not very convenient to use the expansions to higher degrees also. To take care of this fact, we take $\langle \epsilon \rangle$, the average of the above three estimates of ϵ, and use it to find out the value of ν from the equation

$$\nu = \frac{1}{2}\left(1 + \frac{\langle \epsilon \rangle}{6}\right). \tag{3}$$

Thus an estimate of νd is obtained, which can be compared to $\bar{\nu}$ obtained from the finite-size scaling. We see that when $\delta - 1$ is small, the agreement is nice (Table I). Naturally, when δ is considerably away from 1, the effective dimensionality decreases and the epsilon expansions are not very useful here, particularly for α and γ. This is reflected in the mismatch of the estimates from epsilon expansion and the finite-size scaling analysis for $\delta = 1.6$ and $\delta = 1.8$.

V. SUMMARY AND CONCLUSIONS

We have investigated the behavior of the Ising model on a one-dimensional network in which there are links to random long-range neighbors existing with a probability, which varies inversely with the distance along the original chain following Eq. (1). The effective dimensionality of the network deviates from 1 continuously as δ is made smaller than 2 and a finite-temperature phase transition is observed. The transition temperature increases as δ is made smaller and the critical exponents vary with δ as the dimensionality changes. The results show that there is a mean-field behavior for $\delta < 1$ and a finite-dimensional behavior for $1 < \delta < 2$. In the mean-field regime, the finite-size scaling analysis works with an effective dimensionality of four, the upper critical dimension of the Ising model. The present study confirms the conclusion made in Ref. [15] that the small-world behavior of the underlaying network exists for $\delta < 1$ only.

Our results also show that the effective dimensionality cannot be simply extracted from the behavior of the shortest paths on this network although the latter shows a power-law behavior with the number of nodes. We have used an alternative method to estimate the effective dimensionality, which works quite well for δ close to 1.

The present study was conducted to address certain issues related to the topology of a network, but the study of phase transition of the Ising model on networks has its own inherent interest as the Ising model has wide applications in many interdisciplinary areas. For example, the study and nature of phases in such models gives us important insight into social opinion dynamics [33] in closely knit populations, where the networks look very similar to the models discussed here, and opinions are being modeled as states of spin vectors at each site.

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[24] Here one first selects a site, say A, and "burns" its neighbors, which form a set N_1. All neighboring sites of the sites belonging to N_1, which are not yet burned, are burned next and these form the set N_2. The process continues until all the sites are burned. The distance from A to another site B is equal to x if B belongs to set N_x. A description of the burning algorithm can be found in H. J. Herrmann, D. C. Hong, and H. E. Stanley, J. Phys. A **17**, L261 (1984).

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